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On the Superfluid Fraction of an Arbitrary Many-Body System at $T=0$

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I derive a set of sufficient and (barring certain pathologies) necessary conditions for a one-component system with velocity-independent forces to have a superfluid fraction ρ_s/ρ equal to unity at zero temperature: In addition to a condition closely related (but not obviously equivalent) to the usual one of off-diagonal long-range order, the ground state should possess unbroken invariance under both spatial translation and time reversal. Some generalizations are made to the case of multicomponent systems.

KEY WORDS: Superfluidity; long-range order; two-fluid model.

Consider an arbitrary system of neutral particles for which the thermodynamic limit, understood in the usual sense, exists. The *superfluid fraction* of such a system may be defined by the following standard thought-experiment: Place the system in a large, approximately cylindrically symmetrical container of annular or toroidal shape, such that the circumference $2\pi R$ is large compared to the characteristic transverse dimension d and the classical moment of inertia I_{cl} is thus, to zeroth order in d/R, given by $2\pi R^3 A\rho$ ($\equiv NmR^2$ for a single species), where ρ is the average mass density and A is the cross-section of the torus. Now rotate the walls of the container with an angular velocity ω, and require that the system be in thermodynamic equilibrium as viewed from the frame of the rotating walls. The superfluid density $\rho_s(T)$, or equivalently the superfluid fraction ρ_s/ρ, is then defined by the relation

$$f_s(T) \equiv \rho_s(T)/\rho \equiv 1 - \lim_{\omega \to 0} (\langle L \rangle / I_{\text{cl}}\omega) \tag{1}$$

where $\langle L \rangle$ is the expectation value of the mechanical angular momentum.

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Two points should be noted about the definition (1). First, it is of course implicit that the limit exists, and thus that the expectation value of $\langle L \rangle$ with the walls stationary is zero; and we will actually make, below, the rather stronger assumption that time reversal symmetry is not spontaneously broken in this state. Secondly, the definition relates to a state of *thermodynamic equilibrium*; it is emphatically not implied that the fraction of the system which can sustain circulating supercurrents (a thermodynamically *metastable* phenomenon) is necessarily given by the quantity $f_s(T)$. In this connection I mention a rather tricky point relating to the precise meaning of the limit on the right-hand side of (1), and the related thermodynamic limit: we should take $N \to \infty$, d, $R \to \infty$, $\omega \to 0$ in such a way that $d/R \to 0$, $mR^2\omega/\hbar \to 0$ but $NmR^2\omega/\hbar \to \infty$.

Given the definition (1) of the superfluid fraction $f_s(T)$, it is immediately obvious that there exists a large class of systems for which this quantity is zero at all T. Indeed, at present there exist only two neutral (terrestrial) systems which have been actually demonstrated experimentally to have a nonzero value of $f_s(T)$ at any T, namely the two stable isotopes of helium in the liquid phase. (The general belief is that the recently stabilized Bose-condensed phase of the atomic alkali gases will also have a finite superfluid density, but this has not yet been explicitly established). Now, it has been established with fair confidence in the case of ^{4}He, and is at least consistent with experiment (and generally believed) in the case of ^{3}He, that the quantity $f_s(T)$ satisfies the relation

$$\lim_{T \to 0} f_s(T) = 1 \tag{2}$$

or in words, that the whole liquid is superfluid at zero temperature. The aim of the present essay is to discuss the status of Eq. (2) from a theoretical standpoint: that is, to ask the question: What are the necessary and sufficient conditions for Eq. (2) to be true in an arbitrary many-body system? While it is clear that this question is closely related to the issues discussed in some classic papers of the 60's (e.g., those of Yang,[1] Kohn,[2] Kohn and Sherrington[3] and Bloch[4]), I am not aware of any place in the existing literature where it is explicitly discussed in a model-independent way. The present discussion may be regarded as a continuation of the line of thought developed in refs. 5–7.

Before embarking on the general question, let's briefly review some relevant results which can be obtained within specific models or assumptions. First, in the case of a translation-invariant Fermi system which forms Cooper pairs in a way such that the ground state and low-lying excitations are well described by the simple BCS theory, it is straightforward to show that the superfluid fraction is just $1 - Y(T)$ where $Y(T)$ is the so-called

Yosida function; since the latter tends to zero in the limit $T \to 0$, Eq. (2) is automatically fulfilled. This result is independent of the symmetry of the pairing state, and moreover remains true[8] in the model of a "superfluid Fermi liquid," that is a system which forms Cooper pairs on the background of a normal Fermi liquid as described by the Landau theory, even though in this case the general formula for $f_s(T)$ at arbitrary T is more complicated. Secondly, Eq. (2) is obviously trivially true for a noninteracting Bose gas, and Gavoret and Nozières[9] have shown that for an interacting Bose system such as ^{4}He it remains true provided that the system can be described by perturbation theory starting from the free Bose-condensed gas. Thirdly, at first sight, at least, the relation should hold true for any system which has a superfluid phase describable by the standard Landau two-fluid hydrodynamics. The argument[6] goes as follows: if the superfluid velocity $\underline{v}_s(\underline{r}t)$, total mass current density $\underline{j}(rt)$, pressure $P(\underline{r}t)$ and chemical potential $\mu(\underline{r}t)$ are defined in the standard way, then in the long-wavelength, low-frequency limit where the two-fluid hydrodynamics applies we have

$$\frac{d\underline{j}}{dt} = -\underline{\nabla}P, \qquad \frac{d\underline{v}_s}{dt} = -\underline{\nabla}\mu \tag{3}$$

However, in the limit $T \to 0$ the Gibbs–Duhem relation implies that $\underline{\nabla}P = \rho\underline{\nabla}\mu$, from which we have in the limit $\underline{v}_s \to 0 \underline{j} = \rho\underline{v}_s + f(r)$, where f is an arbitrary function of position but is independent of time. If we require that this result agree with the general two-fluid formula $\underline{j} = \rho_s\underline{v}_s + \rho_n\underline{v}_n$ $(\rho_n \equiv \rho - \rho_s)$, we see that f must be identically zero and thus at $T=0$ we have $\rho_s = \rho$, i.e., Eq. (2) is satisfied. However, the apparent generality of this result is spurious, as one can see for example by considering a ^{3}He–^{4}He mixture under the assumption (which may not be true but is at least not obviously internally inconsistent) that the ^{3}He quasiparticles do not form Cooper pairs; in this case we should expect a finite normal density ρ_n at zero temperature, and indeed it turns out that the hydrodynamic "derivation" fails, since the Gibbs–Duhem relation no longer has the simple form involved above. This example shows that Eq. (2) cannot be generic to superfluid systems, even those with overall translation invariance. In any case, to establish a static property by an argument based on dynamics seems anomalous; in fact, it may be argued that the very use of two-fluid hydrodynamics in some sense begs the question.

In the bulk of this paper I shall consider a system of identical particles of mass m and without internal degrees of freedom, interacting via a two-body isotropic velocity-independent potential (generalization to the multi component case will be made at the end); I make no particular assumption about the statistics obeyed by the particles. Thus, if we denote the coordinate of the ith particle by $\underline{r}_i$, the ground state wave function $\Psi_o\{\underline{r}_i\}$ for boundary conditions at rest satisfies the equation

$$\hat{H}\Psi_o\{\underline{r}_i\} = E_o \Psi_o\{r_i\} \tag{4a}$$

$$\hat{H} \equiv \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m} \underline{\nabla}_i^2 + U(\underline{r}_i) \right) + \frac{1}{2} \sum_{i,j=1}^{N} V(|(\underline{r}_i - \underline{r}_j|) \tag{4b}$$

where $U(\underline{r}_i)$ is an appropriate potential describing the confining effect of the container; it is convenient to allow this to have small deviations from exact cylindrical symmetry. A crucial role in the argument below is played by the "single-valuedness boundary condition" (SVBC); to introduce this we re-express the variable $\underline{r}_i$ in terms of an angular variable θ_i and two "transverse" coordinates η_i whose precise definition is unimportant for our purposes. Then, for the walls at rest, the SVBC reads

$$\Psi_o(\theta_1, \eta_1, ..., \theta_i, \eta_i \cdots \theta_N, \eta_N) = \Psi_o(\theta_1, \eta_1 \cdots \theta_i + 2\pi, \eta_i \cdots \theta_N, \eta_N), \quad \forall i \tag{5}$$

i.e., in words, the result of taking any *one* particle once around the annulus, leaving the rest unchanged, should be to recover the original wave function.

Now, there is a standard argument (cf. e.g., ref. 5), which demonstrates that the definition (1) above of the superfluid density is equivalent to the following one: Let $\Psi_{\Delta\varphi}\{r_i\}$ be the wave function which minimizes the expectation value of $\hat{H}$ (Eq. (4a)), but now subject not to the condition (5) but to the modified SVBC

$$\begin{aligned} &\Psi_{\Delta\varphi}(\theta_1 \eta_1 \cdots \theta_i \eta_i \cdots \theta_N \eta_N) \\ &\quad = \exp(i\,\Delta\varphi) \cdot \Psi_{\Delta\varphi}(\theta_1 \eta_1 \cdots \theta_i + 2\pi, \eta_i \cdots \theta_N \eta_N), \quad \forall i \end{aligned} \tag{6}$$

and let $E(\Delta\varphi)$ be the corresponding lowest energy eigenvalue. As explained in ref. 5, a given value of $\Delta\varphi$ corresponds to a value of ω equal to $(\hbar/2\pi m R^2) \cdot \Delta\varphi$. Then the definition (1) at $T=0$ is equivalent to

$$f_s(0) \equiv \lim_{\Delta\varphi \to 0} (4\pi^2 I_{\mathrm{cl}}/N^2\hbar^2)\, \partial^2 E/\partial(\Delta\varphi)^2 \tag{7}$$

—a result which is sometimes expressed by saying that the superfluid density is the "sensitivity to twist of the boundary conditions" or "helicity modulus." It is important for the subsequent argument that, as follows from the remark about the meaning of the limit in (1) and the relationship between $\Delta\varphi$ and ω, the notation "$\lim_{\Delta\varphi \to 0}$" in Eq. (7) should be understood in the sense that $N^{-1} \ll \Delta\varphi \ll 1$. Note that in the special case that $\Psi_{\Delta\varphi}\{r_i\}$ is related to $\Psi_o\{r_i\}$ by a "Galilean shift," i.e., by the relation

$$\Psi_{\Delta\varphi}\{r_i\} = \exp\left(i \sum_i \theta_i \cdot \Delta\varphi/2\pi\right) \Psi_o\{r_i\} \tag{8}$$

the quantity $\partial^2 E/\partial(\Delta\varphi)^2$ is just $N\hbar^2/mR^2$ and thus Eq. (2) is satisfied.

At this point I will assume that the Hamiltonian $\hat{H}$ has the property of invariance under time reversal and that the latter is not spontaneously broken in the ground state. Then quite generally (and independently of the statistics obeyed by the particles) the ground state wave function $\Psi_o\{\underline{r}_i\}$ can be taken real, so that the expectation value of the current operator everywhere vanishes. Given this state of affairs, it is straightforward to obtain various *upper* limits on the zero-temperature superfluid fraction $f_s(0)$ from variational ansätze of the form

$$\Psi_{\Delta\varphi}^{\text{trial}}\{\underline{r}_i\} = \exp(i\Phi\{r_i\}) \cdot \Psi_o\{r_i\} \tag{9}$$

where the real quantity $\Phi\{r_i\}$ must be symmetric with respect to the interchange $i \leftrightarrows j$ (independently of the statistics!) and satisfy the SVBC $\Phi(\theta_i + 2\pi) = \Phi(\theta_i) + \Delta\varphi, \forall\{\underline{r}_i\}$. An ansatz of the form (9) leaves both the external and the interparticle potential energy unchanged, so that the associated increase in energy ΔE_{trial} has the simple form

$$\Delta E_{\text{trial}} = (\hbar^2/2m) \sum_j \int \Psi_0^2\{r_i\} (\underline{\nabla}_j \Phi\{r_i\})^2 \, d\{r_i\} \tag{10}$$

The simplest implementation of the above scheme is the choice

$$\Phi\{\underline{r}\} = \sum_i \varphi(\theta_i) \qquad \varphi(\theta + 2\pi) - \varphi(\theta) = \Delta\varphi$$

which clearly satisfies the above conditions. On minimizing the energy (10) with respect to the form of $\varphi(\theta)$ and inserting into the definition (7), we obtain[5] an upper limit f_s^+ on $f_s(0)$ of the form (cf. Appendix)

$$f_s^+ = \left[\frac{1}{2\pi} \int_o^{2\pi} \frac{d\theta}{\rho(\theta)/\rho_o}\right]^{-1} \tag{11}$$

where $\rho(\theta)$ is the single-particle density averaged over the cross-section of the annulus and ρ_o its average over θ. The inequality, valid for any positive function g, $\langle g \rangle \langle g^{-1} \rangle \geqslant 1$ ensures that $f_s^+ \leqslant 1$, the upper limit being obtained only for $\rho(\theta) = \text{const.}$; we thus see that the condition of uniform (with respect to θ) single-particle density is a necessary, though not sufficient, condition for the validity of Eq. (2).

The problem of obtaining a *lower* limit on $f_s(0)$ is considerably more delicate. Let us *assume* for the moment that the *true* wave function $\Psi_{\Delta\varphi}\{r_i\}$ is of the form (9), so that the corresponding energy is given by the RHS of Eq. (10). We may be able to find a lower limit on this expression, if we can split it up into two terms E_1 and E_2, of which E_2 can be demonstrated to be positive, and minimize E_1 exactly under conditions which are not more restrictive than the actual ones (though they may be less so).

The most obvious implementation of this proposal is to replace the sum over j in (10) by N times one of its members, say k, and to minimize the expression so obtained subject to the single SVBC $\Phi(\theta_k+2\pi)-\Phi(\theta_k)=\Delta\varphi$, $\forall\{r_i\}$ with no restriction of symmetry imposed. (So far, $E_2=0$). In words, we are asking "how much energy does it cost to deform the wave function of the kth particle, at a constant configuration of the other $N-1$ particles, so that the phase accumulated on going around the annulus is $\Delta\varphi$?" If the many-body wave function $\Psi_o\{r_i\}$ is positive definite *everywhere* in the configuration space, a lower limit on the resultant expression can be obtained by now taking E_2 to be the "transverse" part of the kinetic energy in (10), and solving explicitly for $\varphi(\theta_k:\{\xi\})$ (where $\{\xi\}$ indicates *all* coordinates other than θ_k, including the "transverse" coordinates of the kth particle). The result, expressed as a formula for a lower limit f_s^- on $f_s(0)$, is (Appendix)

$$f_s^-=\int\frac{d\xi}{(1/2\pi)\int(d\theta_k)/(\rho_o(\xi,\theta_k))} \tag{12}$$

where $\rho_o(\xi,\theta_k)$ is the square of the (normalized) many-body ground state wave function $\Psi_o(\theta_k,\xi)$. Unfortunately, the expression on the RHS of (12) is likely to be much smaller than 1, because when the particle k traverses the annulus "at constant transverse coordinate," i.e., on a circular path, it is very likely to encounter regions already occupied by other particles where its probability density is small; indeed, if the mutual interaction of the particles includes an infinite repulsive hard core of finite radius, then we should expect that in the thermodynamic limit the RHS of Eq. (12) would be zero, making the limit trivial. However, it follows from the analysis of Section C of the Appendix (with the assumption made there on the Jacobian) that even in this case the answer to the question posed verbally above will be finite (and an N-independent factor times $(\hbar^2/mR^2)(\Delta\varphi)^2$ in the thermodynamic limit) *unless* there exists at least one hypersurface in the many-dimensional (θ,ξ) space which any path $(\theta,\{\xi\})\to(\theta+2\pi,\{\xi\})$ must intersect and on which $\Psi_o(\theta,\xi)$ vanishes everywhere. (Here and in the following, I shall use the phrase "a (nodal) hypersurface exists" as a shorthand for "such a surface either literally exists in the ground state wave function $\Psi_o\{r_i\}$, or can be introduced into it by a deformation which costs an energy which in the thermodynamic limit is at most of order N^{-1}": cf. the usual definition of spontaneously broken symmetry) If such a hypersurface does exist, then the "relative phase" of the many-body wave function at θ and $\theta+2\pi$ is not defined; I return to this point below. For the moment it is enough that the lower limit on $f_s(0)$ calculated from the above "single-particle" argument, while it may not be easy to calculate explicitly, is not identically zero: cf. Appendix.

A more fruitful separation of the terms in (10) is into center-of-mass (COM) and relative coordinates. We introduce in the standard way the "angular COM coordinate" Θ by

$$\Theta \equiv N^{-1} \sum_{i=1}^{N} \theta_i \tag{13}$$

and a set of $3N-1$ other coordinates ξ' whose precise choice is of no consequence: the important point is that the RHS of Eq. (10), (call it E_-) can be written in the form

$$E_- = \frac{\hbar^2}{2mN} \int \Psi_o^2(\Theta, \xi') \left(\frac{\partial \phi}{\partial \Theta}(\theta, \xi')\right)^2 d\Theta\, d\xi' + E_2 \equiv E_1 + E_2 \tag{14}$$

where E_2 is a positive functional of the gradients with respect to ξ'. We minimize the term E_1 with respect to the functional form of $\Phi(\Theta, \xi')$ subject to the (incomplete) SVBC $\Phi(\Theta+2\pi, \{\xi'\}) - \Phi(\Theta\{\xi'\}) = N \cdot \Delta\varphi$, $\forall\{\xi'\}$. The result, expressed in the form of a (prima facie!) lower limit f_s^- on $f_s(0)$, is

$$f_s^- = \int \frac{d\xi'}{(1/2\pi)\int (d\Theta)/(\rho_o(\Theta, \xi'))} \qquad (\rho_o(\Theta, \xi') \equiv \Psi_o^2(\Theta, \xi')) \tag{15}$$

Expression (15) at first sight looks analogous to (12). However, the important difference is that while, as we have seen, for fixed ξ in (12) there may be regions of θ where the quantity $\rho_o(\theta, \xi)$ is small or even zero, in (15) the quantity $\rho_o(\Theta, \xi')$ is very likely to be quite uniform in Θ even for fixed ξ'. In fact, if (a) the Hamiltonian is invariant under rotation around the axis of symmetry (i.e., in the thermodynamic limit, translation-invariant), and (b) this invariance is not spontaneously broken in the ground state, then $\rho_o(\Theta, \xi')$ must be *independent* of Θ for all ξ', and hence the "lower limit" f_s^- becomes unity. Note that this conclusion is completely independent of the statistics obeyed by the particles, or of the details of their interactions.

At first sight this conclusion is alarming, since combined with the result (11), it would appear to imply that *any* translation-invariant many-body system will be superfluid at $T=0$, with a superfluid fraction equal to one. The resolution to this pseudo-paradox lies, of course, in the observation that even if the true wave-function $\Psi_{\Delta\varphi}\{r_i\}$ satisfies Eq. (9) (for the case that it does not see below), the form of the SVBC we imposed in the above argument, namely $\Phi(\Theta+2\pi, \{\xi'\})-\Phi(\Theta, \{\xi'\})=N\cdot\Delta\varphi$, $\forall\{\xi'\}$, is actually more restrictive than the (relevant part of) the true one, which in fact reads

$$\Phi(\Theta+2\pi, \{\xi'\})-\Phi(\Theta, \{\xi'\})=N\,\Delta\varphi(\mathrm{mod}\,2\pi), \quad \forall\{\xi'\} \tag{16}$$

Because our definition of the superfluid fraction (see above) relies on values of $\Delta\varphi$ which, while small compared to 1, are large compared to N^{-1}, the modification introduced in (16) is not at all trivial. What it means is that *as regards the SVBC on the COM coordinate Θ*, we are always free to choose the LHS of Eq. (16) to be $\leqslant\pi$, and the corresponding lower limit (14) is then multiplied by a factor of order N^{-2}, making it indistinguishable from zero for all practical purposes.

Why then are not all many-body systems "normal" at $T=0$ ($f_s(0)=0$)? The answer is of course that while our earlier SVBC was in a sense too restrictive, Eq. (16) is not restrictive enough; we need to go back to the original requirement (6). *If* the "single-particle phase difference" $(\Phi(\theta_k+2\pi, \{\xi\})-\Phi(\theta_k, \{\xi\}))$ is well-defined, then by symmetry it must be equal to $N^{-1}[\Phi(\Theta+2\pi, \{\xi'\}-\Phi(\Theta, \{\xi'\})]$, and hence the RHS of (16) can take *only* the value $N\,\Delta\varphi$ (for $\Delta\varphi\ll 1$); in that case the argument leading to (15) goes through and we indeed conclude that if the system is translation invariant then Eq. (2) holds. Now we saw above that a sufficient condition for the single-particle phase difference to be well-defined was the absence of a "nodal hypersurface" of the kind described. Thus, such absence, plus translation invariance, is a sufficient condition for Eq. (2) to hold. Note that it is irrelevant whether there exist hypersurfaces on which Ψ_o takes a very small value, provided only that this value (call it ε) does not tend to zero in the thermodynamic limit; in particular, ε can be as small as we like and $f_s(0)$ will remain unity! (It is tempting, here, to make a connection with a well-known characteristic of superfluid ^{4}He, namely that provided the "condensate fraction" is finite in the thermodynamic limit, its actual value is irrelevant to the superfluid properties).

The above condition, while sufficient, is of course not necessary: even if the "one-particle" trajectory $(\theta, \{\xi\})\to(\theta+2\pi, \{\xi\})$ crosses a nodal hypersurface, the two-particle trajectory $(\theta_1, \theta_2, \{\xi''\})\to(\theta_1+2\pi, \theta_2+2\pi, \{\xi''\})$ need not do so, and if it does not then the argument for (15) goes through just as in the one-particle case. More generally, we expect (15) to be valid if there exists *any* "low-order" value of n (i.e., n of order 1, not of order N) for which the "order-n trajectory" $(\theta_1, \theta_2 \cdots \theta_n : \{\xi''\}) \to (\theta_1 + 2\pi,\ \theta_2 + 2\pi \cdots \theta_n + 2\pi : \{\xi''\})$ crosses no nodal hypersurfaces. On the contrary, if one or more such hypersurfaces *which are not dictated by symmetry* exists, we can always use it to put in the appropriate arbitrariness of the phase and $f_s(0)$ will be zero (or of order N^{-2}, see above).

All the above analysis of the lower limit on $f_s(0)$ is based on the assumption that the true wave function $\Psi_{\Delta\varphi}\{r_i\}$, which minimizes the energy subject to the boundary condition (8), has the form given in Eq. (9). What if it does not? In this case we can, quite generally, replace the function $\Psi_o\{r_i\}$ in (9) by some other real wave function $\Psi'\{r_i\}$; with this replacement the above arguments go through verbatim (note in particular that there is no term in the energy linear in $\underline{\nabla}\Phi$), with the exception of the fact that there is now an extra term in the quantity $E(\Delta\varphi)$ which is simply the difference between the quantity $\langle \Psi'_o | \hat{H} | \Psi'_o \rangle \equiv E'_o$ and the true ground state energy $E_o \equiv \langle \Psi'_o | \hat{H} | \Psi_o \rangle$. We see in particular that a necessary condition for $f_s(0)$ to be less than unity is that Ψ'_o possess nodal hypersurfaces of the type described above. However, this condition is not sufficient: the term $E'_o - E_o$ leads to a (positive) term in $f_s(0)$ which is proportional to $(\Delta\varphi)^{-2}$, and since quite generally $f_s(0)$ is bounded above by unity and the lower limit allowed on $\Delta\varphi$ is proportional to N^{-1}, the quantity $E'_o - E_o$ must vanish in the limit $N \to \infty$ (actually as I_{cl}^{-1}, cf. Eq. (7)). Thus, in this case, even though the true ground state wave function Ψ_o need not have the relevant nodal hypersurfaces, it must be possible to deform it into a wave function (Ψ'_o) which does have them at a cost of an energy which tends to zero at least as fast as N^{-1} in the thermodynamic limit. Thus, the condition that "a nodal hypersurface exists," if interpreted in the generalized sense defined above, is indeed fulfilled.

It is clear that the criterion of the existence or not of a (non-symmetry-dictated, NSD) nodal hypersurface for the trajectories of order n is closely related to the concept of the existence of off-diagonal long-range order (ODLRO)[1] in the order-n correlation function; at first sight, at least, we might expect that the existence of an NSD nodal hypersurface would imply the absence of ODLRO and vice versa, so that our criterion for (15) to be valid would reduce to the usual criterion for superfluidity,[1] namely the existence of ODLRO in some low-order ($n \sim 1$, not $\sim N$) correlation function. Actually, while there seems no obvious reason to doubt the proposed connection for a boson system (where there are no "symmetry-dictated" nodes), the question becomes rather delicate for the Fermi case. Indeed, even in the apparently trivial case of a free 3D spinless Fermi gas, while it is clear that the system does not possess ODLRO in any low-order correlation function, it is not immediately clear that it automatically possesses the NSD nodal hypersurfaces required by the above argument. We may obtain some clues by considering the (distinctly pathological) case of N free fermions on a 1D ring;[7] this system is known to have $f_s(0) = +1$ when N is odd but a *negative* superfluid fraction, $f_s(0) = -1$, when N is even (a case violating our original condition of no spontaneous breaking of time reversal symmetry in the ground state), and this behavior is associated with the presence, in the latter case but not the former, of an NSD nodal hypersurface. My guess is that the 3D Fermi system manages to produce, at the cost of an energy which tends to zero in the thermodynamic limit (cf. above), NSD nodal hypersurfaces which in some sense average over the above behavior so as to give $f_s(0) = 0$; but the precise way in which this happens is likely to depend strongly on the details of the geometry, and I have not explored it in detail at the time of writing.

To summarize the results of the discussion so far: For a simple one-component system with velocity-independent forces, a set of sufficient conditions for Eq. (2) to hold is that[2]

(1) at least one low-order trajectory fails to intersect any NSD nodal hypersurface, and (C1)

(2) the Hamiltonian is invariant under translation and time reversal, and neither of these invariances is broken in the ground state (C2)

We now examine more briefly to what extent the set of conditions (C1)–(C2) is necessary as well as sufficient. It is clear that condition (C1) is necessary, since if it is not fulfilled we can always put in the required "phase jumps" on the order-1 NSD nodal hypersurfaces, and in fact attain in this way $f_s(0) = 0$ (or more precisely $f_s(0) \leqslant 0$, since we cannot a priori exclude that there may be an even more energetically favorable solution!). As regards (C2), let's first consider the case where translation invariance (but not necessarily time reversal invariance) is either absent in the Hamiltonian (as would be the case, for example, for a substantially cylindrically unsymmetrical container) or spontaneously broken in the ground state. In the latter case by definition, and in the former case automatically except conceivably for very pathological cases, the *many-particle* ground state probability density is a nontrivial function of the COM coordinate Θ for at least some values of the relative coordinates $\{\xi'\}$. (and we can sharpen up the definition of "spontaneously broken symmetry" to ensure that the latter do not form a set of zero measure). Unfortunately, this feature alone appears insufficient for a proof that $f_s(0) < 1$, since any variational ansatz must

[2] One could of course leave out the "NSD;" for the reason it is included, see next paragraph.

respect (5) and not just (15). On the other hand, a variational proof based on (11) requires that the *single-particle* probability density $\rho(\theta)$ be a nontrivial function of θ: see Eq. (12). Is it, then, conceivable that $\rho_o(\Theta, \{\xi'\})$ should be a nontrivial function of Θ while simultaneously $\rho(\theta)$ is constant as a function of θ? At the time of writing I have been unable to find a rigorous and general proof to exclude this possibility, but it seems extraordinarily improbable. If indeed $\rho(\theta)$ is a nontrivial function of θ, while condition (1) and the condition of unbroken time reversal invariance still hold, one would expect that $0 \leqslant f_s < 1$; an example of such a situation, the hypothetical "supersolid" phase of a boson system, was discussed in ref. 5.

Finally, what happens if time reversal is either absent in the Hamiltonian or spontaneously broken in the ground state, while condition (C1) and translation invariance (may) still hold? Here we recall that we are explicitly dealing with systems with no internal degree of freedom such as spin, so that the only relevant effect of the time reversal operation is on the orbital variables. In general, if the ground state lacks time reversal invariance, then the quantity $f_s(0)$ is not even defined, since by a correct choice of the sign of the symmetry breaking and a variational ansatz of the form (9) it will be possible to produce a term linear in $|\Delta\varphi|$ and hence nonanalytic at $\Delta\varphi = 0$. It is clear that a sufficient condition for this situation to hold is that the ground state expectation value $\langle \underset{\sim}{L} \rangle$ of the angular momentum be nonzero. Is this also a necessary condition? i.e., if the ground state breaks time reversal invariance, but in such a way that $\langle \underset{\sim}{L} \rangle = 0$ (as conceivably might happen in a hypothetical system of spinless fermions with p-wave BCS pairing, for example), would $f_s(0)$ necessarily be defined and equal to 1?[3] My instinct is no (i.e., that $f_s(0)$ would not be defined, for the reason given above), but at the time of writing I have been unable to construct a generic proof.

To summarize, for the one-component system with velocity-independent forces discussed so far, (C1) is a necessary condition for Eq. (2) to hold, and (C2) is also necessary if we exclude the kind of semi-pathological situations discussed in the last two paragraphs. We can further state that if translation-invariance is not spontaneously broken in the ground state, then quite generally either $f_s(0) = 1$ or $f_s(0) \leqslant 0$, i.e., the system is either (super-)normal or "completely" superfluid.

It is straightforward to generalize the above considerations to a multicomponent system, bearing in mind that the relation between $\Delta\varphi$ and ω will be different for particles of different mass. When we do so, we find that

[3] It is worthwhile to emphasize that simple application of a standard Galilean transformation does *not* answer this question, any more then it would for the "supersolid."

translation-invariance no longer guarantees that $f_s(0)$ must be either ($\leqslant$) 0 or 1, as in the single-component case; the reason is that since (e.g.) not all N order-1 trajectories are now equivalent, some may intersect NSD nodal hypersurfaces while others do not. If we call the first class A and the second B, then we can apply to the type-B atoms the same argument as we used above to arrive at Eq. (12), while the type-A atoms will give zero contribution to the RHS of Eq. (10). In this way we obtain the result $f_s(0) \leqslant \rho_A/(\rho_A + \rho_B)$, where $\rho_{A,B}$ denotes the mass densities of the two species. This result agrees with the conclusions of analyses based on specific microscopic models, e.g., of dilute solutions of ^{3}He in ^{4}He. (In the latter system, it can be shown that the $T=0$ normal density (i.e., $(1-f_s(0))(\rho_A + \rho_B)$) is equal to the number density of ^{3}He atoms times the "dynamic effective mass;" since the latter is always $\geqslant$ the bare ^{3}He mass,[12] the above result follows).

Note that from the point of view of the present discussion (see particularly last paragraph) any internal degree of freedom such as spin automatically makes the system "multicomponent," so that the conclusions proved in the main body of the text cannot be assumed to hold without further argument. For the case of spin 1/2, *provided* that time reversal invariance applied to the orbital coordinates *alone* is a good symmetry, one may use overall time reversal invariance to argue that the two spin species must behave identically, and thus the pair of conditions (C1)-(C2) above remain both necessary and sufficient for (2) to hold. To see that this extension is not entirely trivial, consider the case of a, spin-1 Fermi system such as is effectively formed[4] by atomic ^{6}Li (see ref. 11). If this system forms Cooper pairs, they may or may not spontaneously break time reversal invariance, but even if they do not it is shown in ref. 11 that the superfluid fraction at zero temperature will be different from unity (this is essentially because one of the three spin species will inevitably be "left out" of the pairing and will thus effectively form a normal component).

In conclusion, in this paper I have shown that for a one-component system with velocity-independent forces there exists a set of sufficient and (barring pathologies) necessary conditions for the superfluid fraction at zero temperature to be equal to unity, namely the conditions (C1)-(C2) above. I have also indicated how to generalize some of the results to a multicomponent system.

[4] Admittedly, in real life the system as considered in ref. 11 could only be stabilized by a high magnetic field, which automatically breaks time reversal invariance. However, there seems no internal inconsistency in a model in which this feature is absent.

APPENDIX

In this appendix I use for the sake of clarity dimensionless variables. However, to emphasize the connection with the (dimensional) variables used in the main text of the paper, I denote the former by the corresponding symbols (ξ, E, etc.)

Consider a system described by a single "longitudinal" coordinate z, where $0 \leqslant z \leqslant 1$, and a set of transverse coordinates which for notational simplicity I label simply ξ. A real, positive, normalized and continuous function $\rho(z, \xi)$ satisfying $\rho(1, \xi) = \rho(0, \xi)$, $\forall \xi$ is specified, and it is required to find the minimum value of the functional

$$E = \int_0^1 dz \int d\xi\, \rho(x, \xi) \left\{ \left(\frac{\partial \varphi}{\partial z}(z, \xi) \right)^2 + (\underline{\nabla}_\xi \varphi(z, \xi))^2 \right\} \tag{A1}$$

over a real function $\varphi(z, \xi)$ satisfying the boundary condition

$$\varphi(1, \xi) = \varphi(0, \xi) + \Delta\varphi, \qquad \forall \xi \tag{A2}$$

In Eq. (1) the notation $(\underline{\nabla}_\xi \varphi)^2$ is simply a schematic representation of the squared gradient with respect to all the ξ-coordinates. The "superfluid fraction" f_s is defined by

$$f_s = \lim_{\Delta\varphi \to 0} \partial^2 E / \partial(\Delta\varphi)^2 \tag{A3}$$

Apart from trivial questions of normalization, etc., it is clear that the problem of finding either the value of f_s, or upper and/or lower limits on it, is precisely that discussed at several different points in the text, with different choices of z and ξ. As noted in ref. 13 (where it is discussed for the special case of ξ 2-dimensional), it is in that case also exactly isomorphic to a problem of classical electrostatics, namely the problem of finding the capacitance of a condenser filled with a material of arbitrarily but continuously varying isotropic dielectric constant (and I should not be at all surprised to find that the theorems proved below are given in some nineteenth-century textbook on that subject, though if so I have so far failed to find it!)

A. Upper Limit on f_s

Defining the quantity $\bar{\rho}(z) \equiv \int \rho(z, \xi)\, d\xi$, we make the variational ansatz

$$\varphi(z, \xi) = \varphi(z), \qquad \partial\varphi/\partial z = \text{const.}\, (\bar{\rho}(z))^{-1} \tag{A4}$$

and substitute in (A1). This gives

$$f_s \leqslant \left\{ \int_0^1 \frac{dz}{\bar{\rho}(z)} \right\}^{-1} \equiv \left\{ \int_0^1 \frac{dz}{\int d\xi\, \rho(z, \xi)} \right\}^{-1} \tag{A5}$$

B. Lower Limit on f_s

It is clear that the term in $(\underline{\nabla}\varphi)^2$ in (A1) is positive, so we can find a lower limit on E, and hence on f_s by finding the exact minimum of the first term. A straightforward use of the standard calculus of variations show that this is achieved by setting

$$\frac{\partial \varphi}{\partial z}(x, \xi) = f(\xi) \cdot (\rho(z, \xi)^{-1} \quad \forall \xi \tag{A6}$$

where the function $f(\xi)$ is chosen so as to satisfy Eq. (A2). Substitution of (A6) into (A2) gives

$$f_s \geqslant \int d\xi \left\{ \int_0^1 \frac{dz}{\rho(z, \xi)} \right\}^{-1} \tag{A7}$$

In the special case of $\rho(z, \xi)$ factorizable, i.e., $\rho(z, \xi) = \bar{\rho}(z)\, g(\xi)$, the limits (A5) and (A7) coincide and thus the exact result is

$$f_s = \left(\int_0^1 dz/\rho(z) \right)^{-1} \qquad (\rho(z\xi) \text{ factorizable} \tag{A8}$$

C. A More General Lower Limit (Schematic)

We can generalize the argument leading to (A7) by considering an *arbitrary* collection of paths which run from $(0, \xi)$ to $(1, \xi)$ for all possible values of ξ, provided that we re-parametrize the space appropriately (in (B) we considered only linear paths parallel to the z-axis). If λ parametrizes the particular path (λ might for example be the value of ξ at which the path starts and ends) and s_λ the distance along it, then schematically we can write the resulting lower limit on f_s in the form

$$f_s \geqslant \int d\lambda \left\{ \int ds_\lambda [K(\lambda, s_\lambda)\, \rho(z(\lambda, s_\lambda), \xi(\lambda, s_\lambda))]^{-1} \right\}^{-1} \tag{A9}$$

where $K(\lambda, s_\lambda)$ is some positive "Jacobian" weighting factor which will depend in detail on the specification of the "trajectories" $z(\lambda, s_\lambda)$, $\xi(\lambda, s_\lambda)$. (e.g., for all trajectories parallel to one another $K(\lambda, s_\lambda)$ is simply unity).

Assuming that it is always possible to choose the trajectories so that $K(\lambda, s_\lambda)$ is everywhere nonzero, then we see that f_s must be greater than zero *unless* every possible path $(0, \xi) \to (1, \xi)$ for every value of ξ crosses a nodal hypersurface. This is the result needed in the text.

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